False Theorems and Fake Proofs

David Grabovsky

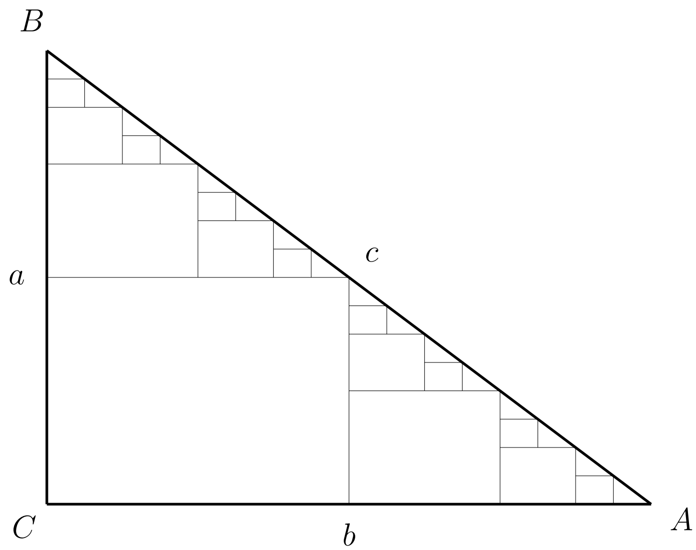
August 4, 2023

We present several elementary results in the burgeoning field of AFC (Alternative Facts, with the Axiom of Choice). Most of these are "theorems" that are obviously false, but with "proofs" that sound surprisingly convincing. For each one, a short explanation of what goes wrong with the proof is provided, but a more complete analysis of what breaks down is left to the reader. Laughter should be the reward for understanding.

Contents

1 Warm-Up: Pythagoras is Drunk

Theorem 1.1 (Pythagoras)**.** *In a right triangle with sides a, b and hypotenuse c, $c = a + b$.*



Proof. We can travel from B to A in two ways: (1) along the hypotenuse c; or (2) along side a and then along side b. The former path has length c, while the latter path has length $a+b$. But we could also take the following jagged route: travel halfway down a, then move parallel to b until we hit the midpoint of side c; then continue downwards, parallel to a, until we hit b, and then move along the latter half of b. This route, too, has total length $a + b$. We can then take successively more jagged routes, as shown above, defined by repeatedly applying the side-bisection procedure. At each stage, the length of the "crinkly" path remains $a + b$, while the path itself becomes closer and closer to c. We obtain an infinite sequence of paths $\{p_n\}_{n=1}^{\infty}$ whose lengths are all $a+b$. In particular, the length of the limiting path p_∞ is $a+b$. But every point on the limiting path p_∞, which consists entirely of corners, lies on c; in other words, the limiting path is c itself, and therefore has length c. Therefore $a + b = c$. ∎

Explanation. In order for a curve to be endowed with a meaningful notion of length, it must be *smooth*, i.e. it must have a well-defined slope at every point. The infinitely crinkly limiting path constructed above fails in this regard, and cannot really be said to have a length at all. In more technical lingo, arclength is defined only for C^1 curves, whose defining functions or parametrizations have at least one continuous derivative. The construction above shows that a limit of piecewise-C^1 curves need not be C^1 anywhere. One can also view this as a purely analytic pathology of some infinite sequences of curves or functions.

2 Some Basic Proof Techniques

2.1 Contradiction: One is Maximal

Theorem 2.1. *One is the largest real number.*

Proof. Suppose, to the contrary, that some $n > 1$ is the largest real number. Then n^2 must be strictly less than n, since n is the largest. (We cannot have $n^2 = n$ because $n \neq 1$). Therefore $n^2 - n < 0$. But we may also write $n^2 - n = n(n-1)$. And because both n and $n-1$ are positive, the product $n(n-1)$ must be positive. This is a contradiction, so our initial assumption must be false. The largest value that does not run afoul of the argument above is $n = 1$, so we conclude that 1 must be the largest real number. ■

Explanation. The "proof" above is actually a valid argument for the *uniqueness* of the largest real number: if it exists, it must be 1. The main fallacy here is the tacit (and moreover incorrect) assumption of the *existence* of a largest real number.

2.2 Induction: All Numbers are Equal

Theorem 2.2 (Marx)**.** *All natural numbers are equal.*

Proof. Let n denote the larger of two any natural numbers a, b: $n = \max\{a, b\}$. We will show by induction on n that $a = b$ for any value n of their maximum. For the base case, when $n = 0$, we have $a = b = 0$ because both a and b are natural numbers. For the inductive step, suppose for some $k = \max\{a, b\}$, that $a = b$. Now take $\max\{a, b\} = k + 1$. We have

$$\max\{a, b\} = k + 1 \implies \max\{a - 1,\ b - 1\} = k, \tag{2.1}$$

so by the inductive hypothesis $a - 1 = b - 1$, and therefore $a = b$. This completes the inductive step and shows that any two natural numbers are equal to the larger among them. In other words, all natural numbers are equal to each other. ■

Corollary 2.3 (Nietzsche)**.** *All natural numbers are equal to zero.*

Proof. This follows directly from Theorem 2.2. But let us give an independent proof. We will show by strong induction on n that all $n \in \mathbb{N}$ are equal to zero. Certainly the base case $0 = 0$ holds. Next, we assume that for *all* $k \leq n$, we have $k = 0$. In particular, $1 = 0$. But then it follows that $k + 1 = 0 + 0 = 0$, which completes the inductive step. ■

Explanation. Both of the arguments above exhibit problems with the base cases of the induction—the "actual proofs" have no real problems. In the first case, the application of the inductive hypothesis to $a - 1$ and $b - 1$ is invalid unless the first *two* base cases ($n = 0$ and $n = 1$) have been checked. Equivalently, the mistake is the invalid claim that $a - 1$ is always a natural number, in particular for $a = 0$. The second argument is more or less the same, as evidenced by the use of $1 = 0$. Interestingly, though, the actual mistake is *not* the act of assuming that all $k \leq n$ are zero: instead, it is the act of writing $k = (k-1)+1$ and then doing induction without checking the first two base cases as before.

2.3 A Teaser: Anomalous Fractions

The following fact is actually true, but its proof are still bogus.

Theorem 2.4. *We have* $\frac{16}{64} = \frac{1}{4}$.

Proof. We apply the cancellation rule for fractions:

$$\frac{16}{64} = \frac{1\not{6}}{\not{6}4} = \frac{1}{4}.$$

■

Explanation. There are three other "anomalous" two-digit fractions:

$$\frac{19}{95} = \frac{1}{5}, \qquad \frac{26}{65} = \frac{2}{5}, \qquad \frac{49}{98} = \frac{4}{8}. \tag{2.2}$$

Can you find the three-digit ones? *Hint:* you can write numbers like 16 or 64 in "digit" form as $10a + 1b$. Anomalous fractions will share digits in the numerator and denominator, and a successful cancellation imposes constraints on the participating digits.

3 Abuses of Basic Algebra

3.1 One is Equal to Two

Theorem 3.1 (al-Khwarizmi)**.** $1 = 2$.

Proof. Let a and b be nonzero. If $a = b$, then we have

$$a^2 = ab \implies a^2 - b^2 = ab - b^2 \implies (a-b)(a+b) = b(a-b) \implies a + b = b. \tag{3.1}$$

Since $a = b$, we obtain $a + b = b + b = 2b = b \implies 1 = 2$. ■

Explanation. This is an absolute classic. We divided by zero—can you see where?

3.2 One Equals Minus One

Theorem 3.2 (Riemann)**.** $1 = -1$.

Proof. Watch this: $1 = \sqrt{(-1)(-1)} = \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = i^2 = -1$. ■

Explanation. Square roots do not factor as $\sqrt{ab} = \sqrt{a}\sqrt{b}$ unless both a and b are non-negative. To understand why on a deeper level, you need to know some complex analysis. I won't explain it here, but what happened above is that we took the wrong branch of the square root. Or, to be more geometric about it, we went through the branch cut on the complex z-plane and ended up on the wrong sheet of the Riemann surface of $w = z^2$.

3.3 All Numbers Equal One

Theorem 3.3 (Euler)**.** *Every nonzero complex number is equal to one.*

Proof. Every nonzero complex number may be represented in polar form, $z = re^{i\theta}$, thanks to Euler's formula. We may rewrite this in the following way:

$$z = re^{i\theta} = \exp(\log r + i\theta) = e^{2\pi i x}, \qquad \text{where} \qquad x = \frac{1}{2\pi i}(\log r + i\theta). \tag{3.2}$$

But then we have $z = e^{2\pi i x} = (e^{2\pi i})^x = 1^x = 1$. Even more directly, for any $y \in \mathbb{C}$,

$$e^y = \exp\Big(2\pi i \cdot \frac{y}{2\pi i}\Big) = (e^{2\pi i})^{y/2\pi i} = 1^{y/2\pi i} = (1^{1/2\pi i})^y = 1^y = 1. \tag{3.3}$$

And since any nonzero complex number can be written as the exponential of some $y \in \mathbb{C}$, we conclude again that every nonzero complex number is equal to 1. ■

Explanation. Generalizing the example above, exponents do not factor as $x^{ab} = (x^a)^b$ unless more stringent conditions on x, a, b are satisfied. Can you find these conditions? The deeper explanation once again lies on the complex plane. Here, we ended up on the wrong branch of the infinite Riemann surface of the logarithm. Whether identities like $e^{xy} = (e^x)^y$ are true depends crucially on the branch-cut structure of the function.

4 "Real" Analysis

4.1 Derivatives Gone Wrong

Theorem 4.1 (Leibniz)**.** $1 = 2$ *(again).*

Proof. We begin with the following basic observations:

$$\begin{aligned} 1^2 &= 1 \cdot 1 = 1; \\ 2^2 &= 2 \cdot 2 = 2 + 2; \\ 3^2 &= 3 \cdot 3 = 3 + 3 + 3, \text{ etc.} \end{aligned} \tag{4.1}$$

By analogy, $x^2 = x \cdot x = \overbrace{x + \cdots + x}^{x \text{ times}}$. Taking the derivative of both sides, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x, \qquad \text{but also} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\Big(\underbrace{x + \cdots + x}_{x \text{ times}}\Big) = \underbrace{1 + \cdots + 1}_{x \text{ times}} = x. \tag{4.2}$$

Hence we find $x = 2x$ for every real x, from which it follows that $1 = 2$. ■

Explanation. It is tempting to argue that "x^2 is just x, taken x times" is nonsensical. This is true, but it's not where the proof goes wrong. The main issue with the second calculation of $\frac{\mathrm{d}}{\mathrm{d}x}(x^2)$ is that, while it correctly differentiates the x's being added together, it fails to differentiate the number of times they are added together, which is also x. To take this into account, one needs the product rule. Applying it correctly yields the expected result:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\underbrace{x + \cdots + x}_{x \text{ times}}\Big) = \Big(\underbrace{1 + \cdots + 1}_{x \text{ times}}\Big) + \underbrace{x + \cdots + x}_{1 \text{ time}} = x + x = 2x. \tag{4.3}$$

4.2 Integrals: The Logarithm

Theorem 4.2. *The logarithm is an even function.*

Proof. The antiderivative of an odd function is even, and $\frac{1}{x}$ is an odd function. ■

Explanation. The antiderivative of $\frac{1}{x}$ is $\log|x|$, not $\log x$, and the former is even due to the presence of the absolute value. Mathematicians in Euler's time did not appreciate the absolute value, leading to a vigorous but ultimately silly debate along these lines.

Theorem 4.3. *All real, positive numbers are equal to one.*

Proof. Recall that the integral of $1/x$ is a logarithm:

$$\int \frac{\mathrm{d}x}{x} = \log|x|. \tag{4.4}$$

We can also compute the integral by writing the integrand as

$$\frac{1}{x} = a \cdot \frac{1}{ax}, \qquad a > 0. \tag{4.5}$$

Now make the change of variables $u = ax$, so that $\mathrm{d}u = a\,\mathrm{d}x$. Then, observe that

$$\frac{\mathrm{d}u}{u} = \frac{a\,\mathrm{d}x}{ax} = \frac{\mathrm{d}x}{x}. \tag{4.6}$$

That is, the integrand is scale-invariant. Overall, we obtain

$$\log|x| = \int \frac{\mathrm{d}x}{x} = \int \frac{\mathrm{d}u}{u} = \log|u| = \log|ax|. \tag{4.7}$$

Therefore we find $|ax| = |x|$, so it holds for every real $a > 0$ that $a = 1$. ■

Explanation. This is a classic case of being careless with the limits of integration. Even though it is true that $\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}u}{u}$, the change of variables in the integral requires a corresponding change of variables in the integration limits, which were conveniently omitted above. Try to put back the limits and ensure that the calculation does not produce a contradiction.

4.3 Integrals: The Sine

Theorem 4.4. *For all $a \in \mathbb{R}_+$ and any integrable function $f\colon [0,a] \to \mathbb{R}$,*

$$\int_0^a f(x)\,\mathrm{d}x = 0. \tag{4.8}$$

Proof. Let us make the substitution $u = \sin\left(\frac{\pi x}{a}\right)$, so that $\mathrm{d}u = \frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)$. Then,

$$\cos\left(\frac{\pi x}{a}\right) = \sqrt{1 - \sin^2\left(\frac{\pi x}{a}\right)} = \sqrt{1-u^2} \implies \mathrm{d}x = \frac{a}{\pi}\frac{\mathrm{d}u}{\sqrt{1-u^2}}. \tag{4.9}$$

Observe that $u = 0$ when $x = 0$ as well as when $x = a$. Our integral becomes

$$\int_0^a f(x)\,\mathrm{d}x = \frac{\pi}{a}\int_0^0 \frac{f\left(\frac{a}{\pi}\sin^{-1} u\right)}{\sqrt{1-u^2}}\,\mathrm{d}u = 0. \tag{4.10}$$

■

Explanation. This one is deeper than it looks. One might object to the handling of the limits of integration, or to the fact that the reparametrization of the interval $[0,a]$ is not one-to-one. Indeed, as x goes from 0 to $\frac{a}{2}$ to a, the u-coordinate goes from 0 to 1 and then back to 0. But folding over the interval isn't the problem. Instead, the issue is that the arcsine function used to solve for x isn't single-valued. The relation $\cos(\pi x/a) = \sqrt{1-u^2}$ should come with a $\pm$, and as a result, the inversion $x = \frac{a}{\pi}\sin^{-1} u$ is only valid for $x \in [0, \frac{a}{2}]$. The integral then needs to be split into two pieces, with x defined correctly in each one, and they will not cancel each other out as the interval is traversed backwards.

A more complex-analytic way of describing the problem is that the function $u = \sin\left(\frac{\pi x}{a}\right)$ has a branch cut, and in traversing the interval from 0 to a on the complex x-plane, one encircles the branch cut on the u-plane and crosses it once. The integral then picks up a nonzero term from the discontinuity across the branch cut. One can also say that the reparametrized interval has a nontrivial winding number and is topologically inequivalent to the original one, or that the integral detects the monodromy picked up by going around the branch point on the Riemann surface of the arcsine function.

5 Advanced Techniques

5.1 An Alternating Series

Theorem 5.1. $1 = 2$ *(one last time).*

Proof. Recall that $\ln(2)$ can be computed from the Taylor series for $\ln(1+x)$:

$$\ln(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \implies \ln(2) = \sum_{i=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{5.1}$$

Let us rearrange the terms of this series. I is not too hard to check that

$$\begin{aligned} \ln(2) &= \sum_{n=1}^{\infty} \left[\left(\frac{1}{2n-1} - \frac{1}{2(2n-1)} \right) - \frac{1}{4n} \right] = \\ &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots = \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2}\ln(2). \end{aligned} \tag{5.2}$$

Hence $\ln(2) = \frac{1}{2}\ln(2)$, and so $1 = \frac{1}{2} \implies 1 = 2$. ■

Explanation. Because this is an infinite series, one needs to pay attention to questions of convergence before actually summing the series. The alternating series of $\ln(2)$ given above is only conditionally convergent, and rearranging infinitely many of its terms can change its value. This is because the series is essentially made up of two interwoven harmonic series, each of which diverge, with terms of opposite signs. The positive and negative parts of the series are each infinite: can you see how to use them to make $\ln(2)$ "equal" anything?

5.2 Fourier Transforms

Theorem 5.2. $1 = 0$.

Proof. Consider the identity function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 1$. Its Fourier transform $\hat{f}$ is the Dirac delta distribution, and is is zero almost everywhere. It follows that the Fourier transform of $\hat{f}$, i.e. $\hat{\hat{f}}$, is identically zero. But $\hat{\hat{f}} = f$ by the Fourier inversion theorem, so we immediately get that f is the zero function $f(x) = 0$. Therefore $1 = 0$. ■

Explanation. That's not how distributions work. If one integrates by parts more carefully, one will see that the Fourier transform of the Dirac delta is a complex exponential, and vice versa. The constant unit and zero functions are both red herrings.

5.3 Countability of the Reals

Theorem 5.3 (Cantor). *The real numbers are countable.*

Proof. We will exhibit a bijection between the set of rational numbers, which is countable, and the set of real numbers. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, one can always find a rational number q_1 between any two real numbers $r_1 < r_2$. Since q_1 is real and distinct from r_1 and r_2, one can find another rational number q_2 between q_1 and r_2. In this way, we have constructed a map between pairs of real numbers and pairs of rational numbers. Take the smaller of each pair; this gives a map $r_1 \mapsto q_1$ from $\mathbb{R}$ to $\mathbb{Q}$. This procedure also works in reverse, since between any two distinct rational numbers there lies a real number (by the completeness of the reals). So the map described above is a bijection, and hence $\mathbb{R}$ is countable. ■

Explanation. This one is essentially complete nonsense—it's a good example of proof by smoke and mirrors. The intent is to intimidate the reader away from asking detailed questions like "can you actually define the map you claim to have constructed?" It's certainly true that between any two rationals is a real number and vice versa, but this does not allow one to define an explicit map (let alone a bijection) between $\mathbb{R}$ and $\mathbb{Q}$. The mapping constructed above is highly non-unique and will generically be neither injective nor surjective. Moreover, the fact that "this procedure also works in reverse" does not imply or do any meaningful work towards a proof that the two maps constructed in either direction are inverses.

6 Epilogue

Theorem 6.1 (Gödel). *There are infinitely many proofs that* $1 = 2$*.*

Proof. We proceed by induction on the number of proofs. Since $1 \neq 2$, the base case $n = 0$ is satisfied. (The case $n = 1$ can be substantiated by any one the examples given above.) Assuming that there are k proofs, we automatically get $k = k + 1 - 1 = k + 2 - 1 = k + 1$ proofs, so the inductive step is complete. This self-referential argument allows one to generate arbitrarily many proofs of this deep and subtle truth as one wants. ■